

# EXPLICIT LIE-POISSON INTEGRATION AND THE EULER EQUATIONS

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**ABSTRACT.** We give a wide class of Lie-Poisson systems for which explicit, Lie-Poisson integrators, preserving all Casimirs, can be constructed. The integrators are extremely simple. Examples are the rigid body, a moment truncation, and a new, fast algorithm for the sine-bracket truncation of the 2D Euler equations.

## §1. INTRODUCTION

Hamiltonian systems are fundamental, and symplectic integrators (SI's) have been increasingly used to do useful extremely-long-time numerical integrations of them. Wisdom [17] has used fast SI's to integrate the solar system far more efficiently than with standard methods; there are numerous examples illustrating the superior preservation of phase-space structures and qualitative dynamics by SI's [9,10,11,14]. Many Hamiltonian systems are not in canonical form but are most naturally written as *Lie-Poisson* systems, which generally arise as reductions from canonical formulations in more variables. Examples are rigid bodies, fluid-particle-field systems (e.g. magnetohydrodynamics or the Vlasov-Poisson equations [7,8]) and general relativity.

Integrating Lie-Poisson PDE's first requires a truncation of the underlying Lie algebra, the most promising approaches at this time being the sine bracket ([5], associated with  $\mathfrak{su}(N)$ ) and applying to systems whose configuration space is the set of symplectic maps of a  $2n$ -dimensional torus, such as 2D incompressible fluids) and, for localized distributions, the moment truncation of Scovel and Weinstein [15]. Then, to integrate in time, there are general methods which provide a symplectic-leaf-preserving Poisson map [4,6]. They are not only implicit but require evaluating functions like " $e^{ad_\epsilon}$ " via Taylor series; hence they can be very slow. Although they can be of any order [2], the time-step must be kept small so that the implicit equations can be solved quickly by iteration—so the implicitness is no advantage.

If Lie-Poisson integrators are to be as practical as standard symplectic integrators they should be simple and fast. To this end we describe the widest general class of such systems for which explicit methods are available. (Examples of such methods were first found by Ruth [13] for canonical systems, and by Channell and Scovel [2] for Lie-Poisson systems.) Our class includes the sine-bracket truncation of the 2D Euler equations—the *sine-Euler* equations—for which the new method is not only explicit but  $\mathcal{O}(N/\log N)$  times faster than the standard implicit method.

## §2. POISSON SYSTEMS & INTEGRATORS

A Lie-Poisson system has a phase space  $M \cong \mathbb{R}^n \ni x$ , a Lie-Poisson bracket  $\{F, G\} = \frac{\partial F}{\partial x_i} J_{ij} \frac{\partial G}{\partial x_j}$  where  $J_{ij} = c_{ij}^k x_k$  (with  $c_{ij}^k$  the structure constants of a Lie algebra) and a Hamiltonian  $H: M \rightarrow \mathbb{R}$ . The dynamics  $\dot{x} = \{x, H\} = J\nabla H$  preserve the Poisson bracket; a Poisson integrator is one whose time-step map  $x \rightarrow x'(x)$  also preserves the Poisson bracket. Symplectic splitting methods apply when the Hamiltonian is a sum of terms each of which can be explicitly integrated—for example, in a canonical Hamiltonian system,  $H = T(p) + V(q)$ , which leads to standard symplectic integrators [13,14]. For Lie-Poisson systems, our methods depend on the following observation. We first form the set of all abelian subalgebras of  $M$ .

**Observation.** *Let*

$$\Sigma = \{\sigma_k \subset \{1, \dots, n\} : J_{ij} = 0 \quad \forall i, j \in \sigma_k\}$$

*Then the Lie-Poisson system with Hamiltonian  $H(\sigma_k)$  (i.e.  $H$  depends only on  $x_i$  with  $i \in \sigma_k$ ) is linear with constant coefficients.*

So if these linear systems can be solved exactly, an explicit first-order Poisson integrator for a Hamiltonian with  $p$  terms  $H = \sum_{k=1}^p H_k(\sigma_k)$  is

$$\varphi = \exp(\Delta t X_1) \dots \exp(\Delta t X_p) \tag{1}$$

where  $\Delta t$  is the time step and  $X_k = J\nabla H_k$ ; that is, just integrate each piece of the Hamiltonian in turn. A second order symmetric method (“leapfrog”) is  $\varphi(\frac{1}{2}\Delta t)\varphi^{-1}(-\frac{1}{2}\Delta t) =$

$$\exp\left(\frac{\Delta t}{2} X_1\right) \dots \exp\left(\frac{\Delta t}{2} X_{p-1}\right) \exp(\Delta t X_p) \exp\left(\frac{\Delta t}{2} X_{p-1}\right) \dots \exp\left(\frac{\Delta t}{2} X_1\right).$$

Methods of any order can be constructed by composing several such steps [2,9,16,18]. Clearly an arbitrary linear term could also be included in any of the  $H_k$ .

Now  $\Sigma$  certainly includes the singleton subsets so we can immediately integrate Hamiltonians of the form  $H = \sum_{k=1}^n H_k(x_k)$ . We choose not to break with tradition and include the rigid body as an example.

**Example: The rigid body.** Here  $\mathbf{m} \in \mathbb{R}^3$  is the angular momentum,  $H = \frac{1}{2}(\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{I_3})$ , and the Lie algebra is  $\mathfrak{so}(3)$  so

$$J = \begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix}.$$

This  $J$  has a Casimir  $C = |\mathbf{m}|^2$  (the total angular momentum) so  $\hat{H} = H - C/2I_1$  has the same dynamics; this leaves only two terms in  $\hat{H}$ . With  $\omega_k = m_k(\frac{1}{I_k} - \frac{1}{I_1})$  and  $R_k(\theta)$  rotation by  $\theta$  around the axis  $m_k$ , the map corresponding to (1) is

$$\mathbf{m}' = R_3(\Delta t \omega_3) R_2(\Delta t \omega_2) \mathbf{m}$$

—a “standard map” of the rigid body. The symplectic leaves  $|\mathbf{m}|^2 = \text{const}$  are clearly preserved. (We believe that this simple method (extended to higher order)

would improve the rigid body simulations of Austin *et al.* [1] who used the (implicit, non-Poisson) midpoint rule.) The same technique applied to the planar pendulum in the form

$$H = \frac{1}{2}x_3^2 + x_2, \quad J = \begin{pmatrix} 0 & 0 & -x_2 \\ 0 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}$$

( $x_1, x_2$  coordinates in the plane,  $x_3$  = angular velocity; Casimir  $x_1^2 + x_2^2$  = length of pendulum) does in fact give the standard map (see, e.g. [4]).

Even if  $H$  is of the form  $\sum H_k(\sigma_k)$ , these methods are only practical if the resulting linear systems can be integrated exactly—diagonalizing them numerically would destroy their efficiency. This has not been a problem in the cases we have done to date: often the linear system is already upper triangular (after a permutation) and can be solved by back-substitution.

**Example: A moment algebra.** Here  $M$  is the algebra of 2nd and 4th order moments of a distribution  $f(q, p)$ . Coordinates are  $\langle q^\alpha p^\beta \rangle \equiv \int q^\alpha p^\beta f(q, p) dq dp$  which we collect in a vector  $x$ . It could arise as a truncation of the evolution of  $f(q, p)$  by Hamiltonian vector fields on the plane, such as the Liouville equation  $\dot{f} + \{f, h\}_{qp} = 0$ , the 2D Euler equations, or the 1D Vlasov-Poisson equations [3,16]. We have

$$x = \begin{pmatrix} \langle q^2 \rangle \\ \langle qp \rangle \\ \langle p^2 \rangle \\ \langle q^4 \rangle \\ \langle q^3 p \rangle \\ \langle q^2 p^2 \rangle \\ \langle qp^3 \rangle \\ \langle p^4 \rangle \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 2x_1 & 4x_2 & 0 & 2x_4 & 4x_5 & 6x_6 & 8x_7 \\ -2x_1 & 0 & 2x_3 & -4x_4 & -2x_5 & 0 & 2x_7 & 4x_8 \\ -4x_2 & -2x_3 & 0 & -8x_5 & -6x_6 & -4x_7 & -2x_8 & 0 \\ 0 & 4x_4 & 8x_5 & 0 & 0 & 0 & 0 & 0 \\ -2x_4 & 2x_5 & 6x_6 & 0 & 0 & 0 & 0 & 0 \\ -4x_5 & 0 & 4x_7 & 0 & 0 & 0 & 0 & 0 \\ -6x_6 & -2x_7 & 2x_8 & 0 & 0 & 0 & 0 & 0 \\ -8x_7 & -4x_8 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

As pointed out in [3], if  $H$  were separable in  $p$  and  $q$  as in the Vlasov-Poisson equations, i.e.  $H = T(x_3, x_8) + V(x_1, x_4)$ , then an explicit Poisson integrator is possible, just as in the canonical case. But from the above observation is it clear that

$$H = H_1(x_1, x_4) + H_2(x_2, x_6) + H_3(x_3, x_8) + H_4(x_4, x_5, x_6, x_7, x_8)$$

also admits an explicit Poisson integrator.

*Extra nonlinear terms*— In the canonical case one often has a nonlinear term in  $H$ , not a function of  $q$  alone, which can nevertheless be integrated explicitly (and conveniently, i.e. without using nonelementary functions). One such is  $H(q_1, q_2^2 + p_2^2)$ , which arises in the nonlinear Schrödinger equation and in the Zakharov equations [9] and which gives rise to the well-known splitting method for the nonlinear Schrödinger equation. This phenomenon is less common in Lie-Poisson systems because of the more complicated evolution of those  $x_j$  not appearing in  $H$ . But it can happen, e.g. let  $H = H(x_1 x_3)$  in the above algebra:

$$\begin{aligned} \dot{x}_1 &= \lambda x_1 & x_1(t) &= e^{\lambda t} x_1(0) \\ \dot{x}_2 &= 0 & \Rightarrow & x_2(t) = x_2(0) \\ \dot{x}_3 &= -\lambda x_3 & x_3(t) &= e^{-\lambda t} x_3(0) \end{aligned}$$

$$\begin{pmatrix} \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \\ \dot{x}_7 \\ \dot{x}_8 \end{pmatrix} = \begin{pmatrix} 0 & 8e^{\lambda t}b & 0 & 0 & 0 \\ -2e^{-\lambda t}a & 0 & 6e^{\lambda t}b & 0 & 0 \\ 0 & -4e^{-\lambda t}a & 0 & 4e^{\lambda t}b & 0 \\ 0 & 0 & -6e^{-\lambda t}a & 0 & 2e^{\lambda t}b \\ 0 & 0 & 0 & -8e^{-\lambda t}a & 0 \end{pmatrix} \begin{pmatrix} x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix}$$

where  $a = x_3(0)$ ,  $b = x_1(0)$ , and  $\lambda = 4x_2(0)H'(ab)$  is constant. This time-dependent linear system can be solved: let  $L(t)$  diagonalize its matrix; then because  $L^{-1}\dot{L}$  is time-independent and tridiagonal, the system is transformed by  $L$  to a time-independent, tridiagonal one, which can be solved by standard methods.

It is not always advantageous to split  $H$  into the fewest possible number of pieces. Splitting even further can lead to less coupled systems to solve and does not affect the error.

The local truncation error of such methods can be easily derived using the Campbell-Baker-Hausdorff formula (see [9]). For a method of order  $l$ , it is a sum of all iterated Poisson brackets of order  $l+1$  of the  $H_k$ 's, with coefficients depending on the order in which the  $H_k$ 's are taken in (1). It may look as if the error "increases with the number of pieces in  $H$ "; but this would mean that the number of ODE's is also increasing, which will affect the error of any method. In fact, numerical simulations show that these methods have roughly the same truncation errors as the implicit Poisson methods. For example, for the rigid body, the above method was  $1.6\times$  less accurate but  $100\times$  faster than the implicit method.

### §3. THE SINE-EULER EQUATIONS

Here we consider the motion of an inviscid incompressible fluid governed by the 2D Euler equations. The field variable is the vorticity  $\omega(x, y)$ , which is  $2\pi$ -periodic in  $x$  and  $y$ . The phase space of vorticities is the dual of the Lie algebra of Hamiltonian vector fields in the plane [8]. We have

$$J = \omega_y \partial_x - \omega_x \partial_y \quad (2)$$

$$\begin{aligned} H &= -\frac{1}{2} \int \psi \omega \, dx dy \quad \text{where } \nabla^2 \psi = -\omega \\ \dot{\omega} &= J(\omega) \frac{\delta H}{\delta \omega} = J(\omega) \psi = -\omega_x \psi_y + \omega_y \psi_x \end{aligned} \quad (3)$$

This algebra has an infinite number of Casimirs, which we may take as

$$C_n = \int \omega^n \, dx \, dy$$

In Fourier space eqs. (2,3) become

$$\begin{aligned} J_{mn} &= \mathbf{m} \times \mathbf{n} \omega_{m+n} \\ H &= \frac{1}{2} \sum_{\mathbf{n} \neq 0} \frac{\omega_{\mathbf{n}} \omega_{-\mathbf{n}}}{|\mathbf{n}|^2} \\ \dot{\omega}_{\mathbf{m}} &= \sum_{\mathbf{n} \neq 0} \frac{\mathbf{m} \times \mathbf{n}}{|\mathbf{n}|^2} \omega_{\mathbf{m}+\mathbf{n}} \omega_{-\mathbf{n}} \end{aligned}$$

where  $\mathbf{m} \times \mathbf{n} = m_1 n_2 - m_2 n_1$  and for real  $\omega$ ,  $\omega_{-\mathbf{n}} = \omega_{\mathbf{n}}^*$ . There is a finite-dimensional truncation of (2) [5], the *sine bracket*

$$J_{\mathbf{mn}} = \frac{1}{\varepsilon} \sin(\varepsilon \mathbf{m} \times \mathbf{n}) \omega_{\mathbf{m+n} \bmod N} \quad (4)$$

where  $\varepsilon = 2\pi/N$  and all indices are henceforth reduced modulo  $N$  to the periodic lattice  $-M \leq m \leq M$  where  $N = 2M+1$ , and we shall take  $N$  prime for convenience (the extensions to nonprime and even  $N$  are straightforward). This  $J$  has  $N-1$  Casimirs which approximate  $C_n$  for  $2 \leq n \leq N$ . Eqs. (4) specify the structure constants of  $\mathfrak{su}(N)$  in an appropriate basis, see [5,19].

The most natural truncated  $H$  is

$$H = \frac{1}{2} \sum_{\substack{n_1, n_2 = -M \\ \mathbf{n} \neq \mathbf{0}}}^M \frac{\omega_{\mathbf{n}} \omega_{-\mathbf{n}}}{|\mathbf{n}|^2}$$

giving the sine-Euler equations, first proposed by Zeitlin [19]:

$$\dot{\omega}_{\mathbf{m}} = \sum_{\substack{n_1, n_2 = -M \\ \mathbf{n} \neq \mathbf{0}}}^M \frac{1}{\varepsilon} \frac{\sin(\varepsilon \mathbf{m} \times \mathbf{n})}{|\mathbf{n}|^2} \omega_{\mathbf{m+n}} \omega_{-\mathbf{n}} \quad (5)$$

where as in (4) all indices are taken modulo  $N$ . As a numerical approximation of (3), these equations are only  $\mathcal{O}(\varepsilon^2)$  accurate. However, once the vorticity has rolled up into small scales, standard (finite difference or spectral) approximations are not very accurate either, and do not possess the Poisson structure or conserved quantities of (5). Perhaps (5) is best regarded as an extremely interesting *model* of the 2D Euler equations until its properties are better understood.

The approach outlined above now gives an  $\mathcal{O}(N^3 \log N)$ , explicit Poisson integrator of (5), preserving all  $N-1$  Casimirs to within round-off error—which is faster than the  $\mathcal{O}(N^4)$  needed just to evaluate the right hand side of (5). We take

$$\sigma_{\mathbf{k}} = \{n\mathbf{k} : 0 \leq n < N\} \in \Sigma.$$

To split the Hamiltonian we need a set of modes that generates the entire lattice, e.g.

$$K = \{(0, 1)\} \cup \{(1, m) : 0 \leq m < N\}$$

(this is why it is convenient to take  $N$  prime—otherwise multiples of  $K$  don't cover the entire lattice) and then

$$H = \sum_{\mathbf{k} \in K} H_{\mathbf{k}}(\sigma_{\mathbf{k}}), \quad H_{\mathbf{k}} = \frac{1}{2} \sum_{n=0}^{N-1} \frac{\omega_{n\mathbf{k}} \omega_{-n\mathbf{k}}}{|n\mathbf{k}|^2}.$$

We now solve the linear ODE's  $\dot{\omega} = J \nabla H_{\mathbf{k}}$ . Of course  $\dot{\omega}_{\mathbf{m}} = 0$  for  $\mathbf{m} \in \sigma_{\mathbf{k}}$ . From (5), the other modes decouple into  $2M$  sets of  $N$  equations, of which we only need to solve  $M$  sets; the others are their complex conjugates (see Figure 1). The variables in each set are a translation of  $\sigma_{\mathbf{k}}$ , say by  $\mathbf{j}$ : let  $z_m = \omega_{\mathbf{j}+m\mathbf{k}}$ , then

$$\dot{z}_m = \sum_{n=-M}^M a_n z_{m-n}$$

where

$$a_n = -\frac{\sin(\varepsilon n \mathbf{j} \times \mathbf{k})}{\varepsilon |n \mathbf{k}|^2} \omega_{n \mathbf{k}}$$

These ODE's are circulant and hence diagonalized by the discrete Fourier transform: let  $\tilde{\mathbf{z}} = F\mathbf{z}$  where  $F$  is the DFT; then

$$\dot{\tilde{\mathbf{z}}} = \Lambda \tilde{\mathbf{z}} \quad \text{where } \Lambda = \text{diag}(F\mathbf{a})$$

so the equations can now be integrated explicitly.

### Summary of algorithm.

- for  $\mathbf{k} \in K$  do
  - for  $\mathbf{j} = 1^{\text{st}}, \dots, M^{\text{th}}$  translation of  $\mathbf{k}$  do [may be done in parallel]
    - with  $z_m = \omega_{\mathbf{j}+m\mathbf{k}}$ , set  $\mathbf{z}' = F^{-1} e^{\Delta t \Lambda} F \mathbf{z}$
    - copy  $(\mathbf{z}')^*$  into  $\omega_{-(\mathbf{j}+m\mathbf{k})}$
  - end do
- end do

The whole procedure requires  $3M(N + 1) = \frac{3}{2}(N^2 - 1)$  DFT's of length  $N$ . As there are FFT's available for sequences of prime length  $N$  [12], the whole algorithm is  $\mathcal{O}(N^3 \log N)$ . Figure 2 shows the relative energy error  $(H(t) - H(0))/H(0)$  for  $10^5$  time steps with  $\Delta t = 0.05$ ,  $N = 7$ ,  $|\boldsymbol{\omega}| = 1$  and  $H(0) = 0.75$ . As is expected from an integrator which is a symplectic map on the symplectic leaves of the phase space, the energy error does not grow with time. The errors in the Casimirs, e.g.  $C_2 = \sum \omega_{\mathbf{n}} \omega_{-\mathbf{n}}$  are due only to roundoff error, and grow by about  $5 \times 10^{-15}$  per time step.

This explicit method exists because the only coupled terms in the Hamiltonian belong to the sets  $\sigma_{\mathbf{k}}$ ; it is fast because of the special form of the sine bracket in this basis. Preliminary simulations indicate that the vorticity does not roll up into the high modes and that the evolution can be followed for arbitrarily long times. It will be interesting to see what the implications of this model are for the ergodicity and statistical steady state of the 2D Euler equations.

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FIGURE 1. Mode splitting in the sine-Euler equations. Here  $N = 7$  and  $M = 3$ .  $\circ$  shows modes in one term  $H_{\mathbf{k}}$  in the Hamiltonian ( $\mathbf{k} = (1, 1)$ ).  $\blacksquare$  shows modes which are coupled together in the linear system  $\dot{\omega} = J \nabla H_{\mathbf{k}}$  (here  $\mathbf{j} = (2, 0)$ );  $\square$  shows modes whose values are the complex conjugate of the  $\blacksquare$  modes.

FIGURE 2. Relative energy error to  $t = 5000$ ,  $\Delta t = 0.05$ ,  $N = 7$ .